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On the nonlinear reflection of a gravity wave at a critical level. Part 2

By S. N. BROWN AND K. STEWARTSON
Department of Mathematics, University College London, London WC1

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In part 1 of this work (Brown & Stewartson 1980*b*) we examined the nonlinear interaction of a forced internal gravity wave in a stratified fluid with its critical level. Although the Richardson number J was taken to be large, the method described there was, in principle, applicable to all Richardson numbers and as such we did not take advantage of the asymptotic properties of the solution of the linearized equations. Here in part 2 we re-develop the linearized solution for a general basic shear and temperature profile when $J \gg 1$ as the large-time limit of an initial-value problem for a wave incident from above the shear layer. On this time scale it is known that the reflection and transmission coefficients are $O(e^{-\nu\pi})$, $\nu = (J-\frac{1}{4})^{\frac{1}{2}}$. It is shown that, when $J \gg 1$, the solution in the neighbourhood of the critical layer consists only of algebraically decaying elements with a direction of propagation parallel to the layer (*critical-level noise*) below a certain level, but of critical-level noise and a wavelike term, corresponding to the imposed incident wave, above this level. On a longer time scale, specifically $t = O(\epsilon^{-\frac{2}{3}})$, where ϵ is the amplitude of the forced wave, the nonlinear terms are no longer negligible; the development of the reflection and transmission coefficients on this time scale is the subject of part 3 (Brown & Stewartson 1982).

1. Introduction

In common with many other unbounded flows a small perturbation to the stream function of a parallel shearing motion of a stably stratified fluid generally consists of two components. Firstly it may contain wavelike elements, generally of the form

$$e^{i\alpha(x-ct)}f(y), \tag{1.1}$$

where α, c are constants, x, y measure distance respectively parallel and normal to the direction of motion of the basic flow, and f satisfies a certain ordinary differential equation. In particular, if there is no shear f is also exponential, and the perturbation is known as an internal wave with a definite phase velocity and direction of propagation. Secondly it may contain algebraic elements of the asymptotic form, when t is large and $t \gg \nu$,

$$t^{\pm i\nu-\frac{3}{2}}\, e^{i\alpha[x-U(y)t]}\, \Psi(y) \tag{1.2}$$

(Booker & Bretherton 1967; Brown & Stewartson 1980*a*), where ν depends on the local Richardson number (supposed greater than $\frac{1}{4}$), $U(y)$ is the local velocity of the fluid and Ψ an arbitrary function of y. Either by inspection or by examining the method of derivation, we can see that (1.2) has the physical interpretation of disturbances being carried along the streamlines of the basic flow. Thus the direction of the energy propagation and of the group velocity is horizontal. Generally these elements correspond to a decaying velocity and may be neglected after a sufficiently long time from the setting up of perturbations and we are left with the more conventional wavelike solution only. However, this is not always the case and our aim in this paper and in part 3 is to describe a situation in which modes like (1.2) play a significant role in controlling the evolution of the disturbance, broadly because $\Psi(y)$ becomes large at one value of y.

We consider a flow field in which the shear is confined to within a finite distance of the plane $y = 0$, but the stratification extends indefinitely both above and below this region. An internal wave is supposed generated at large positive values of y and propagates towards the shear layer. We choose a set of moving axes so that this incident wave once set up can be regarded as static, and examine its long-time interaction with the shear layer in the case when $U(y)$ vanishes at $y = 0$. In physical terms this means that the x-component of the phase velocity of the wave coincides with the shear velocity at one point of the shear layer. We shall also suppose that the local Richardson number is everywhere large; then Booker & Bretherton (1967) have established that on reaching this line, known as the critical level, the wave is absorbed, both the reflected and transmitted waves being exponentially small. We shall examine the nature of this absorption more closely when $\nu \gg 1$ and shall find that there is in fact a transition of the wave from (1.1) to the form (1.2). At first sight this appears surprising since (1.2) decays with time but it is consistent since $f(y) \sim y^{\frac{1}{2}-i\nu}$ while $\Psi(y) \sim y^{-1}$ in a certain sense as $y \to 0$. Hence a transition is feasible when $yt \sim 1$. Strictly the transition level occurs at $\alpha U'(0)\,yt = \nu$, when $t \gg \nu$ and takes place over a distance $O(\nu^{\frac{1}{2}}/t)$. The form (1.1) appears only when $\alpha U'(0)\,yt > \nu$, while the form (1.2) with $\Psi \sim t(\alpha U'(0)\,yt-\nu)^{-1}$ appears on both sides, the singularity being smoothed out in the transition zone. There is, however, a pronounced decrease in the amplitude of the disturbance at this transition, it being smaller by a factor $\nu^{-\frac{1}{2}}$ ahead of the wave front. This absorption may be interpreted as the piling up of the disturbance behind the wave front (at $\alpha U'(0)\,y = \nu/t$), which is moving ever more slowly towards the critical level, reaching it at an infinite time after the forcing started. In addition, there is some conversion of the structure into the other eigensolution corresponding to wave propagation along the shear layer with the local stream velocity. As $t \to \infty$ these two types of solution are almost comparable in size in the critical layer, the second being smaller than the first by a factor $\nu^{-\frac{1}{2}}$, and we shall refer to this second type as *critical-layer noise* (CL-noise). As $|y|$ increases, however, the relative size of the two types falls to $O(\nu t^{-\frac{1}{2}})$ and the CL-noise becomes indistinguishable from disturbances of the form (1.2) generated from other sources such as initial effects; we shall refer to such disturbances as *noise*. For moderate values of ν we may expect that an increased portion of the disturbance is converted into CL-noise and, when ν is small enough ($e^{\nu\pi} \sim 1$), that there will be some reflection and transmission from the linearized theory.

It is now natural to enquire whether a reverse transition can occur and whether noise can be converted into a propagating wave of the form (1.1), which would have to be either a reflected or transmitted wave. Such a question would only have significance for the critical-layer noise, which is of a larger order of magnitude than the background noise and may be distinguished from it. We have already shown in part I (Brown & Stewartson 1980*b*) that a nonlinear wave can be reflected from the critical level, but the method adopted enabled us to construct only the leading term in the amplitude of this wave and did not explain how it is generated. Furthermore, no evidence was found for a transmitted wave, even allowing for nonlinear effects.

We develop here a more detailed account of the structure of the nonlinear evolution of the wave interaction near the critical level and demonstrate first of all that if attention is confined to that part of the wave defined by (1.1) there can be no additional waves generated. This can only come about by considering the nonlinear interaction of (1.1) with the CL-noise defined by (1.2) or even of (1.2) with itself. At each stage of the expansion these nonlinear interactions among the terms already calculated can be regarded as distributed sources for the new term. In general, this term will be of a similar form to the forcing term and hence will be interpretable as CL-noise and will ultimately merge in with the background noise for $|\eta| \gg 1$, where $\eta = \alpha U'(0)yt/\nu$. The reason is that any additional terms would be eigensolutions (or complementary functions) of the linearized equations for small disturbances near $y = 0$. They would have to satisfy regularity conditions at $y = 0$ and must not be representable as inward-moving waves when $|\eta| \gg 1$. These conditions are sufficient to exclude them. An exception occurs if the forced term is singular at one value, η_0 say, of η, for the singularity can be removed by adding an eigenfunction in either $\eta > \eta_0$ or $\eta < \eta_0$ without its violating these conditions. The singularity in the forced term at $\eta = \eta_0$ is defined here to be *resonance*. We note that it is a different phenomenon from the more conventional resonant wave–wave interaction (e.g. Phillips 1966) in that it occurs at one value of η only, rather than for all η. The first resonance appears at the third stage of the expansion and occurs at $\eta = \frac{1}{4}(1+\sqrt{5})^2$, and leads to an eigensolution in $\eta > \eta_0$ only. For large η this has the form of a reflected wave with the same wavelength as the incident wave and the amplitude is computed to be the same as that obtained in the previous study. Continuing, another appears at the fourth stage, occurring at $\eta = 2+\sqrt{3}$, and for large η may be interpreted as a reflected wave of half the wavelength of the incident wave, i.e. the first harmonic. It becomes clear that as the expansion is continued an infinite number of resonances will be generated, leading to a reflected wave containing all the harmonics of the primary wave. The situation is similar in fact to the studies of nonlinear critical layers in geostrophic flows by Stewartson (1978), Brown & Stewartson (1978), Warn & Warn (1976, 1978), Béland (1976), in which it was found that an incident wave generated all the harmonics as a result of nonlinear interactions in the critical region. Then, however, the relative simplicity of the governing equations enabled much greater information to be extracted and even a description of the flow field in the final stages of the nonlinear evolution to be obtained.

A resonance leading to an eigensolution in $\eta < \frac{1}{3}(4+\sqrt{7})$ only ($\eta > \frac{1}{3}(4+\sqrt{7})$ being excluded because it would correspond to an incident wave) occurs at the fourth stage and is a first harmonic, but it remains of the form (1.2) even when η is large and negative. Thus it is not a transmitted wave. Nevertheless, as the expansion is continued this eigensolution continues to interact with the CL-noise from the incident wave in $\eta < 1$ and eventually at the ninth stage in the expansion a resonance appears at a negative value of η which produces a transmitted wave: otherwise the regularity condition at $\eta = 0$ would be violated. The first such wave we have found has the same wavelength as the incident wave but it seems clear that on continuing the expansion all the harmonics will ultimately be generated.

We conclude that, for $\nu \gg 1$, when nonlinear effects are taken into account, the critical layer is not a total absorber of the incident wave but acts on it in a very subtle way, eventually returning some of it first as a reflected wave and then, much later on, as a transmitted wave.

The plan of parts 2 and 3 of this work is as follows. In this paper we concentrate on setting up the basic structure of the linearized disturbance in the shear layer in a form suitable for the computation of the nonlinear disturbance. Part of this structure is associated with the initial perturbation applied (i.e. at $t = 0$) and depends crucially on its properties. Thus, if at large times the nonlinear disturbance can be calculated completely to a particular order of magnitude only (say as far as the square of the amplitude of the initial disturbance) by including such structure, then there is at present little point in carrying out this task, since the result contains a large measure of arbitrariness. We shall show, in § 3, that the CL-noise is independent of such initial perturbations, being fixed by the permanent source of the wave motion outside the shear layer and dependent on its large-time behaviour. Thus it is legitimate to investigate the interaction between the incident wave and this part of the noise. Finally, we examine the structure of the solution in the critical layer with the aim of obtaining an explicit form when $\nu \gg 1$ that is easier to manipulate than that obtained in part 1. There we obtained it as an Hadamard infinite integral that is exact for all ν and, provided it can be handled appropriately, is suitable for computing reflection and transmission coefficients. However, as found there, the details of the computation rapidly became unmanageable. Now we relax the requirement that the linearized solution be exact and obtain instead the leading term of an asymptotic expansion valid when $\nu \gg 1$ so that the nonlinear terms may be calculated explicitly and much more information obtained about the properties of the reflection and transmission coefficients. This task is carried out in part 3 (Brown & Stewartson 1982), and for an explanation in broad terms of the steps in that part of the argument we refer the reader to § 1 of that paper.

2. The basic equations

The physical situation is exactly that of part 1, with an inviscid shear layer separating two parallel streams of fluid in motion, the velocity and density gradient in each stream being uniform but different. We choose orthogonal Cartesian axes Ox^*y^* with origin at some point in the shear layer; Ox^* is parallel to the direction of the streams and O is moving along the x^* axis with the local fluid velocity. We non-dimensionalize the physical variables with a reference speed V^*, length L^* and temperature T_0^* as appropriate, and write the stream function and temperature as basic states together with perturbations $\epsilon\psi(x,y,t)$, $\epsilon T(x,y,t)$, where ϵ is an arbitrary constant and t is the time. Then if, as in part 1, the equation of state is taken to be linear and the Boussinesq approximation is applied, the appropriate equations

$$\left.\begin{aligned}\left(\frac{\partial}{\partial t}+U(y)\frac{\partial}{\partial x}\right)\nabla^2\psi-U''(y)\frac{\partial\psi}{\partial x}+J\frac{\partial T}{\partial x}&=\epsilon\frac{\partial(\psi,\nabla^2\psi)}{\partial(x,y)},\\ \left(\frac{\partial}{\partial t}+U(y)\frac{\partial}{\partial x}\right)T-R'(y)\frac{\partial\psi}{\partial x}&=\epsilon\frac{\partial(\psi,T)}{\partial(x,y)}.\end{aligned}\right\}\qquad(2.1)$$

In (2.1), $U(y)$, $-R'(y)$ are the undisturbed velocity and density in the shear layer, and J is a representative Richardson number with

$$J = \beta^* T_0^* g^* L^* V^{*-2}, \tag{2.2}$$

g^* being the acceleration due to gravity and β^* the coefficient of volume expansion. A fundamental assumption in this paper is that $J \gg 1$.

In part 1 we went on to choose a hyperbolic-tangent velocity profile and a special form of the density gradient in order that the linear steady equations, with $\epsilon = 0$, possessed an analytic solution for all J, which was then examined for $J \gg 1$. The purpose of this was to inspire confidence in the complicated procedure we then applied to the nonlinear terms, some of which were exponentially small in J. However, once the principles there established are accepted, there is an advantage in working with $J \gg 1$ *ab initio*, in which case it is possible to leave $U(y)$, $R(y)$ unrestricted except for the requirement that $U'(y)$ and $R'(y)$ be everywhere positive, $U(0) = 0$, and $U(y)$ and $R'(y)$ tend to limits as $|y| \to \infty$. The constant ϵ in (2.1) will later be taken as small and identified with the amplitude of the incoming plane wave.

Also in part 1 we summarized the results of Booker & Bretherton (1967) on the direction of propagation of plane-wave solutions of (2.1) with $\epsilon = 0$ and $|y| \gg 1$. There we took $U(\infty) = -U(-\infty) = R'(\pm\infty) = 1$, but the generalization is immediate.

Outside the shear layer we are assuming a quasi-steady disturbance in which for $y \gg 1$ there is a forced wave of given amplitude incident on the shear layer from above. This is given by a solution of (2.1) with $\epsilon = 0$ which has an exponential factor

$$e^{i(\alpha x + my)}, \quad U^2(\infty)(\alpha^2 + m^2) = JR'(\infty) \quad (\alpha, m > 0). \tag{2.3}$$

On entering the shear layer the exponential dependence of this wave on y is lost, and eventually the wave is partly absorbed by the critical layer near $y = 0$ and, as we shall show in part 3, is partly reflected or transmitted as waves of the same type with amplitudes depending algebraically on t. In addition, there will ultimately be generated higher harmonics of the primary disturbance, and hence the reflected wave for large positive values of y will be the sum of solutions of (2.1) with exponential factors

$$e^{i(n\alpha x + m_n y)}, \quad U^2(\infty)(n^2\alpha^2 + m_n^2) = JR'(\infty) \quad (m_n > 0), \tag{2.4}$$

$n = 1, 2, \ldots$. Any transmitted wave must also contain higher harmonics, in terms of α, and when y is large and negative will be the sum of solutions of (2.1) with exponential factors as in (2.4) except that $U^2(-\infty)(n^2\alpha^2 + m_n^2) = JR'(-\infty)$. On the other hand, any wave of the form (2.4) when y is large and negative, but having $m_n < 0$, represents a wave incident on the shear layer from below and must be excluded. It should be noted that, for sufficiently large n, m_n becomes imaginary and the wave is evanescent.

We have formulated the problem here for a stably stratified fluid with $R'(y) > 0$, and for convenience have taken $U'(y) \geqslant 0$, $U'(0) \neq 0$, with the wave incident from above on the shear layer. If instead the wave is incident from below the shear layer the solution may be deduced from that discussed here on replacing x, y, T, $U(y)$, $R'(y)$ by $-x$, $-y$, $-T$, $-U(-y)$, $R'(-y)$. This is because, firstly, (2.1) are unaltered by the transformation and, secondly, (2.3) becomes a wave incident below the shear layer, while (2.4) is now a reflected wave for $y \ll -1$ and a transmitted wave for $y \gg 1$. If, however, $U'(y) < 0$ we replace y, $U(y)$, $R'(y)$ by $-y$, $U(-y)$, $R'(-y)$ and set $m < 0$ in (2.3), (2.4).

3. The initiation of the incident wave

In §2 we discussed the wavelike solutions of (2.1) when $|y| \gg 1$ and $\epsilon = 0$, but, granted the exponential dependence on x, $e^{i\alpha x}$, there is another class of solution that corresponds to the algebraic modes of the type (1.2) for which it was necessary that $U'(y) \neq 0$. In fact, when $t \gg 1$ and $y \gg 1$ (2.1) also possesses a solution whose leading term is proportional to

$$|yt|^{-\frac{1}{4}} \exp[i\alpha x - i\alpha U(\infty)t \pm 2i|yt|^{\frac{1}{2}}(R'(\infty)\alpha^2 J)^{\frac{1}{4}}]. \tag{3.1}$$

If $y < 0$ we replace ∞ by $-\infty$. We need to know whether such solutions are likely to be of importance to the evolution of the waves or whether they can be regarded as indistinguishable from the general background noise produced by the way the wave is initiated. This point may be explored by supposing that the wave motion is generated by a forced disturbance proportional to $e^{i\alpha x}$ at $y = y_0$, where y_0 is a large positive number, the disturbance rising smoothly from zero at $t = 0$ to reach a limiting value as $t \to \infty$. Then the governing equations (2.1) may, with $\epsilon = 0$, be solved by a Laplace transform with parameter s on the assumption that ψ and $\partial\psi/\partial t$ are zero at $t = 0$ for $y < y_0$. We take J large and require that $\psi \to 0$ as $y \to -\infty$, i.e. we neglect the possibility that a reflected wave may be generated in the shear layer, and, by using the WKBJ method, obtain

$$\psi(x,y,t) \approx \frac{e^{i\alpha x}}{2\pi i}\int_{c-i\infty}^{c+i\infty} ds\, F(s)\, e^{st}\left(\frac{s+i\alpha U(y)}{s+i\alpha U(y_0)}\right)^{\frac{1}{2}}\left(\frac{R'(y_0)}{R(y)}\right)^{\frac{1}{4}}\exp\left[-\int_y^{y_0}\frac{\alpha(JR'(y_1))^{\frac{1}{2}}}{s+i\alpha U(y_1)}dy_1\right], \tag{3.2}$$

where $F(s)$ is a given function of s, tending to zero very rapidly as $|s| \to \infty$ and such that $sF(s)$ tends to a non-zero limit as $s \to 0$. Apart from these properties the precise form of $F(s)$ depends on the way in which the forced disturbance is set up at $y = y_0$.

Let us first examine the steady component of ψ, namely the contribution to ψ from (3.2) that is obtained from the simple pole at $s = 0$. We define the large positive number ν by

$$\nu^2 = -\tfrac{1}{4} + JR'(0)/U'^2(0), \tag{3.3}$$

and this component may then be written $e^{i\alpha x}\phi(y)$, where

$$\phi(y) = A(y)\, e^{i\nu B(y)}, \tag{3.4}$$

and, if $y > 0$,

$$B'(y) = -\frac{U'(0)}{U(y)}\left(\frac{R'(y)}{R'(0)}\right)^{\frac{1}{2}}, \quad A(y) = A_0\left(\frac{U(y)}{U(y_0)}\right)^{\frac{1}{2}}\left(\frac{R'(y_0)}{R'(y)}\right)^{\frac{1}{4}}\exp\left[-i\nu y_0\frac{U'(0)}{U(\infty)}\left(\frac{R'(\infty)}{R'(0)}\right)^{\frac{1}{2}}\right], \tag{3.5}$$

where A_0 is a constant proportional to $\lim sF(s)$ $(s \to 0)$. This formula was first given by Grimshaw (1976). Strictly, A is the leading term of an asymptotic expansion in descending powers of ν whose coefficients are determinate in terms of A, B. The form for B follows by integration and, since y_0 is large, we may replace y_0 by infinity in any integral which then converges, and write

$$B(y) = \frac{U'(0)}{(R'(0))^{\frac{1}{2}}}\int_y^{\infty}\left[\frac{(R'(y_1))^{\frac{1}{2}}}{U(y_1)} - \frac{(R'(\infty))^{\frac{1}{2}}}{U(\infty)}\right]dy_1 + (y_0 - y)\left(\frac{R'(\infty)}{R'(0)}\right)^{\frac{1}{2}}\frac{U'(0)}{U(\infty)}. \tag{3.6}$$

When $y \gg 1$

$$\phi(y) \approx A_0 \exp\left[-i\nu y\left(\frac{R'(\infty)}{R'(0)}\right)^{\frac{1}{2}}\frac{U'(0)}{U(\infty)}\right], \tag{3.7}$$

in agreement with (2.3) since $J \gg 1$, and as $y \to 0^+$

$$\phi(y) \approx A_0 \left(\frac{R'(\infty)}{R'(0)}\right)^{\frac{1}{4}} \left(\frac{U'(0)}{U(\infty)}\right)^{\frac{1}{2}} y^{\frac{1}{2}-i\nu} e^{-i\Gamma^+\nu}, \tag{3.8}$$

where
$$\Gamma^\pm = \frac{U'(0)}{(R'(0))^{\frac{1}{2}}} \int_0^\infty \left\{ \frac{(R'(0))^{\frac{1}{2}}}{U'(0)\, y_1(y_1+1)} \pm \frac{(R'(\pm\infty))^{\frac{1}{2}}}{U(\pm\infty)} \mp \frac{(R'(\pm y_1))^{\frac{1}{2}}}{U(\pm y_1)} \right\} dy_1. \tag{3.9}$$

The constant Γ^- will be required in §4 and is defined here for convenience. Thus we have recovered the well-known structure of a wave passing through a shear layer as it approaches the critical level $y = 0$. When y is small and negative the form for ϕ can most easily be computed by taking s real and small: it then follows that the integral with respect to y_1 acquires a contribution $\nu\pi$ as y_1 passes through zero from above, and so

$$\phi(y) \approx A_0 \left(\frac{R'(\infty)}{R'(0)}\right)^{\frac{1}{4}} \left(\frac{U'(0)}{U(\infty)}\right)^{\frac{1}{2}} |y|^{\frac{1}{2}-i\nu} e^{-i\Gamma^+\nu} e^{-\nu\pi}. \tag{3.10}$$

This is in agreement with Booker & Bretherton's result that an incident wave suffers a diminution of amplitude by a factor $e^{-\nu\pi}$ on passing through the critical level and henceforth is negligible on our theory. In part 1 results analogous to (3.8), (3.10) were obtained for all ν when $U(y) = \tanh y$ and $R'(y) = 1$, and may be seen to reduce to (3.8), (3.10) as $\nu \to \infty$. The appropriate values of $\Gamma^\pm$ in that case are both $\log 2$.

It might have been expected that (3.2) would have also yielded eigensolutions of the form (1.1) with $c = O(J^{\frac{1}{2}})$ and outside the range of $U(y)$ as studied by Banks, Drazin & Zaturska (1976) and Drazin, Zaturska & Banks (1979). However, in all the examples that they quote, either y is bounded or $R'(y) \to 0$ as $|y| \to \infty$, in which cases our (3.2) is inappropriate. For the problem considered here we suspect that no such eigensolutions exist in the limit $y_0 \to \infty$.

It is clear from (3.8), (3.10) that the limit $t \to \infty$ leads to a singularity in ϕ at $y = 0$ and special care must be taken in the neighbourhood of $y = 0$ when $t \gg 1$. Before discussing this in some detail let us examine the contributions to ψ from the neighbourhoods of the branch points at $-i\alpha U(y)$ and $-i\alpha U(y_0)$. A straightforward calculation shows that the first of these takes the form

$$\left(\frac{R'(\infty)}{R'(y)}\right)^{\frac{1}{4}} \frac{\bar{\nu} F(-i\alpha U(y))}{[2\pi\alpha(U(\infty) - U(y))]^{\frac{1}{2}}} t^{i\bar{\nu}-\frac{3}{2}} \exp\left[i\alpha(x - U(y)\, t)\right] H(y,t), \tag{3.11}$$

where H is an asymptotic expansion in descending integral powers of t whose coefficients are functions of y and $\log t$, the leading term of which is independent of t and has modulus unity, and $\bar{\nu} = (JR'(y))^{\frac{1}{2}}/U'(y)$. Thus the form of ψ when t is large also includes one of the family of algebraic eigenfunctions discussed in §1. However, in general this eigensolution corresponds to a decaying velocity distribution as $t \to \infty$ and moreover depends crucially on the details of the properties of $F(s)$. We refer to the contribution that (3.11) makes to the ultimate form of ψ as *noise* and conclude that it cannot be distinguished from the general background noise inherent in problems of the kind we are studying here. These eigensolutions could arise from any disturbance at any time and so it seems unprofitable to study them any further. There is an exception to this however. When y is very small the eigenfunction (3.11) depends on the behaviour of $F(s)$ when s is small, and develops a formal singularity as $y \to 0$, being proportional to

$$A_0 y^{-1} t^{i\nu-\frac{3}{2}} \exp\left[i\alpha(x - U'(0)\, yt)\right] H(y,t). \tag{3.12}$$

Thus the algebraic eigenfunction near $y = 0$ is special. It is fed not by the initial form of the disturbance at $y = y_0$ but by its steady-state form. Moreover, it is an order of magnitude larger than the general background disturbance and so we refer to it as *critical-level noise*. Finally, at any time t it is comparable with or greater than the wavelike form when $|yt| \ll 1$ and so it cannot be ignored in the all-important region where the shear flow absorbs the incident wave and which, as we shall see in part 3, dominates the nonlinear development of the original disturbance.

The contribution to (3.2) from the neighbourhood of the branch point at

$$s = -i\alpha U(y_0)$$

has a form similar to (3.4) except that $U(y)$ is replaced by $U(y_0) - U(y)$ and A_0 in (3.5) is a function of t such that $A_0 \propto t^{-\frac{1}{2}}$. Hence the contribution to ψ tends to zero uniformly as $t \to \infty$ in the shear layer and may safely be neglected.

It is clear that (3.11) is inadequate when $U(y) = U(\infty)$. In this situation the appropriate form of the eigenfunction is as in (3.1). In fact the derivative with respect to t of the expression in (3.1) with y replaced by $y_0 - y$ may be obtained from (3.2) by taking U, R' to be constant and examining the contribution to (3.2) when $t \gg 1$ from the saddle points at $s = s_\pm$, where

$$s_\pm = -i\alpha U(\infty) \pm i(y_0 - y)^{\frac{1}{2}} (\alpha^2 J R'(\infty))^{\frac{1}{2}} t^{-\frac{1}{2}}. \tag{3.13}$$

4. The reflection and transmission coefficients

In §3 we studied an initial-value problem with forcing at a large finite value of y. The purpose of this was to show that in general there are two types of eigensolutions when $t \gg 1$, one wavelike and another to which we referred as noise. When $|yt| \gg 1$ the noise is negligible in comparison with the wave, but in the neighbourhood of $y = 0$ this is not so and the resulting critical-level noise turns out to be crucial in returning wave energy to the region $|y| = O(1)$. This is a nonlinear effect and will be discussed in part 3. There we shall need the further solutions of the linear steady forms of (2.1) which are subsequently generated, at times $t = O(\epsilon^{-\frac{2}{3}})$, by the match with the nonlinear solution that holds in the critical layer. These were exponentially small when J is large in the initial-value problem of §3 as only the linear terms were retained even near $y = 0$, but as we shall subsequently require them we shall derive them here. They are the solutions that correspond to reflected and transmitted waves as discussed in §2, and will have the behaviour described there when $|y| \gg 1$. In (2.1), with $\partial/\partial t$ and ϵ set equal to zero we write, with n a positive integer to allow for higher harmonics as required,

$$\psi = e^{ni\alpha x}\psi_n(y, t) + \text{c.c.}, \tag{4.1}$$

where
$$U^2(y)\left(\frac{\partial^2 \psi_n}{\partial y^2} - n^2\alpha^2\psi_n\right) - U(y)\,U''(y)\,\psi_n + JR'(y)\,\psi_n = 0 \tag{4.2}$$

and c.c. denotes the complex conjugate. We now seek solutions of (4.2) of the form

$$\psi_n(y, t) = A_{n1}(y, t)\, e^{i\nu B_1(y)} + A_{n2}(y, t)\, e^{i\nu B_2(y)}, \tag{4.3}$$

where A_{n1}, A_{n2} are functions of y and slowly varying functions of t, and ν is defined in (3.3) (again $\nu \gg 1$). There will also be algebraic eigensolutions for ψ of the form (3.11), but the resulting noise is significant only in the neighbourhood of $y = 0$ where they must be retained. However, here where $y = O(1)$ we do not require their precise form so do not calculate them. The functions A_{n1}, A_{n2} will be expressible in series in descending powers of ν and it is easily found from the coefficient of ν^2 in (4.2) that

$$B_1'(y) = \left(\frac{R'(y)}{R'(0)}\right)^{\frac{1}{2}} \frac{U'(0)}{U(y)} = -B_2'(y), \tag{4.4}$$

which we integrate as

$$B_1(y) = \frac{U'(0)}{(R'(0))^{\frac{1}{2}}} \int_{\pm\infty}^{y} \left\{ \frac{(R'(y_1))^{\frac{1}{2}}}{U(y_1)} - \frac{(R'(\pm\infty))^{\frac{1}{2}}}{U(\pm\infty)} \right\} dy_1 + \left(\frac{R'(\pm\infty)}{R'(0)}\right)^{\frac{1}{2}} \frac{U'(0)}{U(\pm\infty)} y, \tag{4.5}$$

according as $y \gtrless 0$, with $B_2 = -B_1$. We note that B_2 corresponds to B in (3.6). The coefficient of ν in (4.2) then shows that, to leading order in ν, $A_{n1} \propto |B_1'|^{-\frac{1}{2}}$ and $A_{n2} \propto |B_2'|^{-\frac{1}{2}}$, and we take

$$\frac{A_{n1}(y,t)}{a_{n1}^{\pm}(t)} = \frac{A_{n2}(y,t)}{a_{n2}^{\pm}(t)} = \left(\frac{R'(\pm\infty)}{R'(y)}\right)^{\frac{1}{4}} \left(\frac{U(y)}{U(\pm\infty)}\right)^{\frac{1}{2}}, \tag{4.6}$$

according as $y \gtrless 0$, where $a_{n1}^{\pm}$, $a_{n2}^{\pm}$ are arbitrary slowly varying functions of t. Thus, when $y \gg 1$,

$$\psi_n(y,t) \approx a_{n1}^{+}(t)\, e^{im_+ y} + a_{n2}^{+}(t)\, e^{-im_+ y} \quad (\alpha m_+ > 0), \tag{4.7}$$

since from (2.3) for fixed α and $\nu \gg 1$ we have

$$U^2(\infty)\, m_+^2 = \nu^2 U'^2(0)\, R'(\infty)/R'(0),$$

and for $y \ll -1$

$$\psi_n(y,t) \approx a_{n1}^{-}(t)\, e^{-im_- y} + a_{n2}^{-}(t)\, e^{im_- y} \quad (\alpha m_- > 0), \tag{4.8}$$

where $U^2(-\infty)\, m_-^2 = \nu^2 U'^2(0)\, R'(-\infty)/R'(0)$. When $|y| \ll 1$ the corresponding forms for $\psi_n(y,t)$ are

$$\psi_n(y,t) \approx \left(\frac{R'(\infty)}{R'(0)}\right)^{\frac{1}{4}} \left(\frac{U'(0)}{U(\infty)}\right)^{\frac{1}{2}} \{a_{n1}^{+}(t)\, y^{\frac{1}{2}+i\nu} e^{i\Gamma^+\nu} + a_{n2}^{+}(t)\, y^{\frac{1}{2}-i\nu} e^{-i\Gamma^+\nu}\}, \tag{4.9}$$

when $y > 0$, and

$$\psi_n(y,t) \approx \left(\frac{R'(-\infty)}{R'(0)}\right)^{\frac{1}{4}} \left|\frac{U'(0)}{U(-\infty)}\right|^{\frac{1}{2}} \{a_{n1}^{-}(t)\, |y|^{\frac{1}{2}+i\nu} e^{i\Gamma^-\nu} + a_{n2}^{-}(t)\, |y|^{\frac{1}{2}-i\nu} e^{-i\Gamma^-\nu}\}, \tag{4.10}$$

when $y < 0$. Here $\Gamma^{\pm}$ are the constants defined in (3.9).

The relevance of (4.6) to (4.10) is as follows. We are going to assume that, when $t \gg 1$, far from the shear layer there is maintained an imposed incident wave of the form $e^{i\alpha x}\phi(y)$, where ϕ is defined in (3.7). The way in which such a wave can develop as t increases is discussed in §3. This wave is represented by the second term in (4.7) with $n = 1$ and a constant value for $a_{12}^{+}(t)$. The first term in (4.7) represents a wave reflected above the critical layer, and in (4.8) the second term represents a wave transmitted through the layer. In §3 we showed that when t is of order unity the transmitted wave is exponentially small in ν and in part 1 we showed that this is also true for the reflected wave. The reason that we are displaying these solutions here is that when t is no longer of order unity, specifically when $t = O(\epsilon^{-\frac{2}{3}})$, ϵ being the amplitude of the imposed incident wave, they are generated by the nonlinear terms that must be retained in the neighbourhood of $y = 0$. The functions a_{n1}^{+}, a_{n2}^{-} develop as functions of τ ($\equiv \epsilon^{\frac{2}{3}}\alpha t$); a_{n1}^{-} must be zero for all n and τ, as it implies the presence of a wave incident from below the shear layer, and $a_{n2}^{+} = 1$ if $n = 1$ and $\tau = 0$ and zero otherwise, since the imposed wave is to remain of constant amplitude. If we define $\mathscr{R}_n(\tau)$, $\mathscr{T}_n(\tau)$ to be the reflection and transmission coefficients associated with the harmonic $e^{ni\alpha x}$ then we shall have

$$\left.\begin{aligned} \psi_n(y,t) &\approx \mathscr{R}_n(\tau)\, e^{im_+y} + \delta_{n1}\, e^{-im_+y} \quad \text{as} \quad y \to \infty, \\ \psi_n(y,t) &\approx \mathscr{T}_n(\tau)\, e^{im_-y} \quad \text{as} \quad y \to -\infty, \end{aligned}\right\} \tag{4.11}$$

and from (4.9), (4.10) correspondingly, as $y \to 0$,

$$\psi_n(y,t) \approx \left(\frac{R'(\infty)}{R'(0)}\right)^{\frac{1}{4}} \left(\frac{U'(0)}{U(\infty)}\right)^{\frac{1}{2}} \{\mathscr{R}_n(\tau)\, y^{\frac{1}{2}+i\nu} e^{i\Gamma^+\nu} + \delta_{n1} y^{\frac{1}{2}-i\nu} e^{-i\Gamma^+\nu}\} \tag{4.12}$$

as $y \to 0^+$, and

$$\psi_n(y,t) \approx \left(\frac{R'(-\infty)}{R'(0)}\right)^{\frac{1}{4}} \left|\frac{U'(0)}{U(-\infty)}\right|^{\frac{1}{2}} \mathscr{T}_n(\tau)\, |y|^{\frac{1}{2}-i\nu} e^{-i\Gamma^-\nu} \tag{4.13}$$

as $y \to 0^-$.

We shall show in part 3 that $\mathscr{R}_n(\tau)$, $\mathscr{T}_n(\tau)$ are of the form

$$\mathscr{R}_n(\tau) = \sum_{r=n}^{\infty} \mathscr{R}_{rn}(\tau), \quad \mathscr{T}_n(\tau) = \sum_{r=n}^{\infty} \mathscr{T}_{rn}(\tau), \tag{4.14}$$

where $\mathscr{R}_{rn}$, $\mathscr{T}_{rn}$ are of the form $\tau^{\frac{3}{2}(r-1)}$ times a function of $\tau^{i\nu}$ and $\epsilon^{i\nu}$. In part 1 we showed that $\mathscr{R}_{11}$, $\mathscr{T}_{11}$, which are independent of τ, are exponentially small in ν. The first non-zero functions are $\mathscr{R}_{31}$, $\mathscr{R}_{42}$, which we find in part 3 for $\nu \gg 1$ and note that $\mathscr{T}_{31}$ and $\mathscr{T}_{42}$ are exponentially small in ν. It emerges that $\mathscr{R}_{31} = O(\tau^{3+2i\nu}/\nu)$ and $\mathscr{R}_{42} = O(\tau^{2i\nu+\frac{9}{2}}/\nu^{\frac{3}{2}})$. Both $\mathscr{R}_{31}$ and $\mathscr{T}_{31}$ were also found in part 1. The first non-zero transmission coefficient turns out to be $\mathscr{T}_{91}$.

In part 1 the results analogous to (4.12), (4.13) were obtained for all ν for

$$U(y) = \tanh y \quad \text{and} \quad R'(y) = 1,$$

and may be seen to reduce to (4.12), (4.13) when $\nu \gg 1$ with an error that is $O(e^{-\nu\pi})$. Since the error in (4.3) is in the functions A_{n1}, A_{n2}, which can be expressed as series in inverse powers of ν, of which (4.6) gives the first terms, and not in B_1, B_2, we expect that the splitting of the incident and transmitted waves as evident in (4.12), (4.13) holds with an error that is again exponentially small in ν. This splitting is of great import for the analysis in the critical layer, for it means that to the critical layer a term of the form $y^{\frac{1}{2}+i\nu}$ for $y > 0$ implies a reflected wave, $y^{\frac{1}{2}-i\nu}$ represents the incident wave and, for $y < 0$, $|y|^{\frac{1}{2}-i\nu}$ implies a transmitted wave and $|y|^{\frac{1}{2}+i\nu}$ must not occur. The form of the exponentially small error cannot be derived from the asymptotic analysis presented here.

As in part 1 the reflection and transmission coefficients are determined by the match with the solution that holds in the neighbourhood of $y = 0$. In §5 the properties of the linearized solution that holds in the critical layer are examined when $\nu \gg 1$.

5. The linearized theory in the neighbourhood of the critical layer

From §§3 and 4 it is clear that special care must be taken in the neighbourhood of $y = 0$ to elucidate the role of the algebraic eigenfunction and its relation to the critical level absorption of the wave. In this section we find the form of the linearized solution near $y = 0$ that corresponds to an outer solution of the form (4.1) with $n = 1$ and $\psi_1(y,t) = \psi_{11}(y)$ independent of t. Thus when $|y| \gg 1$ the outer solution consists of the prescribed incident wave, and possible reflected and transmitted waves as in (4.11), with $\delta_{11} = 1$ and constant reflection and transmission coefficients $\mathscr{R}_{11}$, $\mathscr{T}_{11}$; when $|y| \ll 1$ it takes the forms (4.12), (4.13). The solution near $y = 0$ can be obtained from (3.2), which was derived for a particular set of initial conditions, or from the exact solution in part 1 in the limit $\nu \to \infty$. However, it is instructive to re-derive it by an asymptotic approach, as this is the technique to be used in the nonlinear analysis. The results can then be shown to be consistent with those of other methods and the results of §3. In (2.1) we set $\epsilon = 0$ and $\psi = e^{i\alpha x}\,\Phi(y,t)/U'(0) + \text{c.c.}$ and replace $U(y)$, $R'(y)$ by their leading-order terms in the critical layer with $\partial/\partial y \gg \partial/\partial x$. Then Φ satisfies

$$\left(\frac{\partial}{\partial t} + i\alpha U'(0)\,y\right)^2 \frac{\partial^2 \Phi}{\partial y^2} - \alpha^2 U'^2(0)\,(\nu^2 + \tfrac{1}{4})\,\Phi = 0, \tag{5.1}$$

and may be written as the sum $\Phi = \Phi_- + \Phi_+$, where Φ_-, Φ_+ satisfy

$$\left(\frac{\partial}{\partial t} + i\alpha U'(0)\,y\right)\frac{\partial \Phi_-}{\partial y} - i\alpha U'(0)\,(\tfrac{1}{2} - i\nu)\,\Phi_- = 0, \tag{5.2}$$

$$\left(\frac{\partial}{\partial t} + i\alpha U'(0)\,y\right)\frac{\partial \Phi_+}{\partial y} - i\alpha U'(0)\,(\tfrac{1}{2} + i\nu)\,\Phi_+ = 0, \tag{5.3}$$

since the operator on Φ in (5.1) is the product of the commuting operators on $\Phi_\pm$ in (5.2), (5.3). The advantage of this additive split of Φ is that, since a possible solution for Φ_- has $\Phi_- \propto |y|^{\frac{1}{2}-i\nu}$ as $|y| \to \infty$, and a possible solution for Φ_+ has $\Phi_+ \propto |y|^{\frac{1}{2}+i\nu}$ as $|y| \to \infty$, it is, as may be seen from (4.12), (4.13), Φ_- that will match with the incident wave and the transmitted wave (if any) outside the critical layer, and Φ_+ that will match with the reflected wave.

In part 1, the solution of (5.1) was obtained for all ν by taking a Laplace transform in t, and the functions $\Phi_\pm$ expressed as integrals from which it may easily be seen that $\Phi_\pm$ are of similarity form; with hindsight we therefore write

$$\Phi_\pm = (\alpha t)^{-\frac{1}{2} \mp i\nu} f_\pm(\eta), \quad \eta = \alpha U'(0)\,yt/\nu, \tag{5.4}$$

the powers of t being determined by the fact that Φ_- is to match with the incident wave in (4.12) and the transmitted wave in (4.13) as $|\eta| \to \infty$, and Φ_+ is to match with the reflected wave in (4.12). On substitution into (5.2), (5.3) we find that

$$\frac{\eta}{\nu} f''_\pm + \left[i(\eta \mp 1) + \frac{1}{2\nu}\right] f'_\pm - i(\tfrac{1}{2} \pm i\nu) f_\pm = 0. \tag{5.5}$$

We consider the equations for f_- and, using the WKBJ method, obtain a solution for $\nu \gg 1$ in the form

$$f_-(\eta) = c_{-1}\,|\eta|^{\frac{1}{2}}\,e^{-i\nu \log|\eta|} + c_{-2}\,e^{-i\nu\eta}/(\eta - 1), \tag{5.6}$$

where c_{-1}, c_{-2} are constants. The first term here is actually a multiple of $\eta^{\frac{1}{2}-i\nu}$ and is an exact solution corresponding to the exact solution $y^{\frac{1}{2}-i\nu}$ of (5.2). This solution is clearly unacceptable in any region including the origin. The second term of (5.6) is the leading term of an asymptotic expansion in descending powers of ν, the coefficients of which are similar to that of the leading term. The singularity at $\eta = 1$ is not a property of the basic equation (5.5) for f_- but is a feature of the asymptotic analysis and can be smoothed out in a boundary layer at $\eta = 1$. To achieve this we write

$$\zeta = (\eta-1)(\tfrac{1}{2}\nu)^{\frac{1}{2}}, \quad f_-(\eta) = e^{-i\nu\eta} g_-(\zeta), \tag{5.7}$$

so that to leading order g_- satisfies

$$\tfrac{1}{2}g''_- - i\zeta g'_- - ig_- = 0, \tag{5.8}$$

with solution

$$g_-(\zeta) = \alpha_- e^{i\zeta^2} \int_{-\infty}^{\zeta} e^{-i\zeta_1^2} d\zeta_1 + \beta_- e^{i\zeta^2}, \tag{5.9}$$

where α_- and β_- are constants.

We are now in possession of the solution in the three regions $\eta > 1$, $\eta < 1$ and $\eta - 1 = O(\nu^{-\frac{1}{2}})$. Let us write

$$f_-(\eta) \approx a_- \eta^{\frac{1}{2}-i\nu} + b_- e^{-i\nu\eta}/(\eta-1) \quad \text{when} \quad \eta > 1, \tag{5.10}$$

$$f_-(\eta) \approx d_- e^{-i\nu\eta}/(\eta-1) \quad \text{when} \quad \eta < 1, \tag{5.11}$$

where a_-, b_-, d_- are constants, and take f_- as given by (5.7), (5.9) when $\eta - 1 = O(\nu^{-\frac{1}{2}})$. The six constants $a_-, b_-, d_-, \alpha_-, \beta_-, \mathscr{T}_{11}$ may now be found by matching these solutions together and using the imposed conditions as $\eta \to \infty$. From the match as $\eta \to 1^-$ and $\eta \to -\infty$ we have

$$\beta_- = 0, \quad d_- = i\alpha_-(2\nu)^{-\frac{1}{2}}, \tag{5.12}$$

and from the match as $\eta \to 1^+$ and $\zeta \to +\infty$ we have

$$b_- = d_-, \quad a_- = \alpha_- \pi^{\frac{1}{2}} e^{-i\nu - \frac{1}{4}i\pi}. \tag{5.13}$$

The match of $f_-(\eta)$ as $\eta \to +\infty$ with the incident-wave contribution to ψ_{11}, as given by (4.12), shows that

$$a_- = \left(\frac{R'(\infty)}{R'(0)}\right)^{\frac{1}{4}} \frac{(U'(0))^{1+i\nu}}{(U(\infty))^{\frac{1}{2}}} e^{-i\Gamma^+\nu} \nu^{\frac{1}{2}-i\nu}, \tag{5.14}$$

and that between f_- and ψ_{11}, as given by (4.13) as $\eta \to -\infty$, then shows that

$$\mathscr{T}_{11} = 0. \tag{5.15}$$

The solution for f_+ may be obtained in a similar way. The boundary layer is now at $\eta = -1$ but, as there is no forcing, all the corresponding constants, including $\mathscr{R}_{11}$, are zero, so that f_+ itself is zero.

In part 1 it was shown that $\mathscr{R}_{11}$, $\mathscr{T}_{11}$ are both exponentially small and $O(e^{-\nu\pi})$ for $\nu \gg 1$. The method of matched asymptotic expansions employed here will not yield these exponentially small terms but we may infer that they are both smaller than any negative power of ν from the fact that formally (5.10), (5.11) may be made exact by replacing b_-, d_- by series in inverse powers of ν, but a_- is unaffected. A similar comment applies to the corresponding terms of f_+. None of these algebraic terms will give a contribution to $\mathscr{R}_{11}$, $\mathscr{T}_{11}$.

The solution that we have derived in this section requires comment. We have shown that, when $\alpha U'(0) yt > \nu$ and $\nu \gg 1$,

$$\psi = e^{i\alpha x} \frac{a_-}{U'(0)} (\alpha t)^{-\frac{1}{2}+i\nu} \left\{ \eta^{\frac{1}{2}-i\nu} + (2\pi\nu)^{-\frac{1}{2}} e^{i\nu + \frac{3}{4}i\pi} \frac{e^{-i\nu\eta}}{\eta - 1} \right\} + \text{c.c.}, \tag{5.16}$$

where a_- is given by (5.14), but that, when $\alpha U'(0)\,yt < \nu$,

$$\psi = e^{i\alpha x}\frac{a_-}{U'(0)}(\alpha t)^{-\frac{1}{2}+i\nu}\,e^{i\nu+\frac{3}{4}i\pi}\frac{e^{-i\nu\eta}}{\eta-1}+\text{c.c.}, \tag{5.17}$$

these two solutions being related by a boundary layer at $\eta = 1$ of thickness $O(\nu^{-\frac{1}{2}})$. The incident wave is evident as the first term of (5.16) and the second term is the critical-layer noise described in §3. As shown there, the coefficient of this term is in fact independent of the initial conditions.

The corresponding form for T is, in each region,

$$T = \frac{i}{\nu}\frac{R'(0)}{U'(0)}\frac{\partial\psi}{\partial y}, \tag{5.18}$$

with a relative error $O(\nu^{-1})$.

The alternative way of deriving (5.16), (5.17) is to use the exact solutions of (5.2), (5.3), which are proportional to

$$*\int_0^{\alpha t}\frac{e^{-iU'(0)yu}}{u^{\frac{3}{2}\mp i\nu}}\,du, \tag{5.19}$$

taking the upper sign for (5.2) and the lower sign for (5.3), and examine the behaviour of the integrals by the method of steepest descent. In part 1 the problem was worked with a multiple of the exact form of the integral with the upper sign for $\psi e^{-i\alpha x}$, it being shown that there, as here, the coefficient of that with the lower sign is exponentially small.

A physical interpretation of the solution we have obtained in this section and in §4 may be made as follows. The forcing at $y = \infty$ outside the shear layer generates a stream function $O(\epsilon)$ and velocity components (q_x, q_y) along and perpendicular to the shear layer with $q_x = O(\epsilon\nu)$ and $q_y = O(\epsilon)$ when $y = O(1)$; the disturbance at large time is dominated by the incoming wave. Since q_x must be small we note that an additional condition is that $\epsilon\nu \ll 1$. However, as y decreases the structure of the wave is modified by the shear, and q_x increases although q_y decreases. Just behind the wave front at $y = \nu(\alpha U'(0)\,t)^{-1}$, where $t \gg \nu$, the stream function is proportional to $\epsilon y^{\frac{1}{2}-i\nu}$ and the tangential velocity q_x is $O(\epsilon(\nu t)^{\frac{1}{2}})$, though the normal velocity q_y is only $O(\epsilon(\nu/t)^{\frac{1}{2}})$. In addition, noise is generated at the shear layer but generally its amplitude decays with t, the associated tangential and normal velocity components being $O(\epsilon\nu t^{-\frac{1}{2}})$ and $O(\epsilon\nu t^{-\frac{3}{2}})$, respectively. Near the wave front they increase rapidly in size, however, and when $\eta = O(1)$ are $O(\epsilon t^{\frac{1}{2}})$ and $O(\epsilon t^{-\frac{1}{2}})$ (see (5.16)). As $\eta \to 1$ the velocity components associated with the CL-noise are increased further by a factor $O(\nu^{\frac{1}{2}})$, the two terms of (5.16) become of the same order of magnitude, and the noise contribution is comparable with that due to the incoming wave. When $\eta < 1$ only these components are present and the CL-noise can be interpreted as the precursor of the wave front. In a region $\eta - 1 = O(\nu^{-\frac{1}{2}})$ the magnitude of the perturbation is reduced abruptly by a factor $O(\nu^{-\frac{1}{2}})$, and the hitherto-dominant wave is entirely replaced by a disturbance propagating in the x-direction with the local mean speed of the fluid.

The structure of ψ near $\eta = 1$ is classical for wave fronts in a dispersive medium, the transition function g_- in (5.9) being a Fresnel integral. Examples abound, and an interesting one occurs in the theory of lee-wave trains in rotating fluids (McIntyre 1972).

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